One Relation Says It All

The study of conic sections represents a significant portion of high school mathematics. The equations of circles, parabolas, ellipses, and hyperbolas are explored in considerable detail in many different forms. In some cases a particular form will be well suited for a particular purpose, and in other cases a particular form is only a different form with no special application or property.

What do all of these forms have in common? One answer, of course, is that they are all conic sections. Another is that they can all be expressed as different forms of just one mathematical relation—the General Quadratic Relation.

The **General Quadratic Relation** is:

$$\mathbf{A}x^2 + \mathbf{B}xy + \mathbf{C}y^2 + \mathbf{D}x + \mathbf{E}y + \mathbf{F} = 0 \text{ when A, B, and C are not all 0.}$$

Choose the proper values of A, B, C, D, E, and F and you can express *any* circle, parabola, ellipse, or hyperbola. This expression is sometimes called the **General Quadratic Equation.** Either name is correct. This relation is worth a closer look.

The closer look you'll find in many textbooks looks something like this:

> Given a General Quadratic Relation in which $B \neq 0$, we can rotate the coordinate axes through an angle θ, when $\cot 2\theta = \frac{A - C}{B}$, to make the axes parallel to the axis (parabola) or axes (ellipse and hyperbola) of symmetry of the conic section represented by the relation.
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> This allows us to rewrite the General Quadratic Relation with no xy-term (i.e., B' = 0) as:
>
> $$A'(x')^2 + C'(y')^2 + D'x' + E'y' + F' = 0.$$
>
> Note that A' and x' indicate the new values of A and x after the axes are rotated. The same is true for C', D', E', F', and y'.
>
> The x and y coefficients of the new equation can be found from the equations:
>
> $$x = x' \cos\theta - y' \sin\theta \text{ and } y = x' \sin\theta + y' \cos\theta.$$

Every statement in this description is valid. However, if the description is simply stated without further explanation, I consider it to be **mathemagic** at its best. Good grief! Where did any of this stuff come from? How do we know the value given for angle θ is correct? How do we know the coefficient B' will become 0? How do we find x' and y' from the equations given, and where did the equations come from? How do we find the value of A', C', D', E', and F'? These are the kinds of questions that you should raise when you see a statement of "facts" such as those in this example. Mathematicians should always be asking the simple questions, "Why? How do you know that's true?" The answers to these and other questions regarding the description above are answered in the next section.